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SOME SOLUTIONS OF THE SOLUTE DIFFUSION EQUATION FOR AN ESSENTIALLY VERTICALLY HOMOGENEOUS ESTUARY

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In an earlier report on tidal mixing in estuaries [1]* an equation for the diffusion of a solute in an essentially vertically homogeneous estuary in which the major influences affecting the movement and mixing of fluid are the tides and river flow is derived, namely

$$c \frac{\partial s}{\partial t} + R \frac{\partial s}{\partial x} = D \frac{\partial}{\partial x}(x^2 \frac{\partial s}{\partial x}) + q(x,t), \qquad (1)$$

where c is the time average of the estuary cross-section over a tidal cycle, assumed here to be independent of both x and t;

R is the river discharge and has dimensions L^3T^{-1};

D is a positive constant having dimensions L^2T^{-1} and is defined in Eq. 116 of reference [1];

q is the amount of solute added externally and has dimensions $ML^{-1}T^{-1}$;

s is the concentration of the solute, having dimensions ML^{-3};

x is the distance along the estuary, measured downstream from the position at which the motion in the estuary due to the tides is assumed to be zero.

In this report we consider several solutions of (1); in the first s is the concentration of salinity in the estuary, R is a function of time and $q(x,t) = 0$; in those following, s is the concentration of an externally introduced solute, R is constant and q is assumed to depend only on x.

We proceed then with the solution of

$$c \frac{\partial s}{\partial t} + R \frac{\partial s}{\partial x} = D \frac{\partial}{\partial x}(x^2 \frac{\partial s}{\partial x}) \qquad (2)$$

in which $R = R(t)$, and D and $(a + b)$ are constants. It is assumed that the time variation of R is sufficiently small so that $(a + b) \frac{\partial s}{\partial t}$ is small relative to the other terms in (2) and hence we use the procedure of successive approximations in the

* Numbers in square brackets refer to the bibliography at the end of the paper.

solution of (2).

The boundary conditions to be placed on each of the successive approximations of s are that the salinity at the mouth of the estuary be σ, a constant, and that the salinity at some point upstream be zero. Since the river discharge is a function of time we wish to permit the distance, L, from this point to the mouth (the effective length of the estuary) to vary with time, in which case it is convenient to choose our coordinate system with origin at the mouth of the estuary. However, with this shift in the origin the term x^2 in (2), which is proportional to the square of the distance moved by a particle due to the tides, must be replaced by $(L + x)^2$ so that we now have

$$c \frac{\partial s}{\partial t} + R \frac{\partial s}{\partial x} = D \frac{\partial}{\partial x}\left[(L + x)^2 \frac{\partial s}{\partial x}\right] , \tag{3}$$

in which R and L are assumed to be known functions of time. It must also be assumed that R and L do not vary significantly over a tidal period in order that the derivation which leads to equation (1) remain valid. Physically, this appears to be a reasonable assumption.

We now define successive approximations s_0, s_1, ... to the solution of (3) by

$$D \frac{\partial}{\partial x}\left[(L + x)^2 \frac{\partial s_0}{\partial x}\right] = R \frac{\partial s_0}{\partial x} \tag{4}$$

$$D \frac{\partial}{\partial x}\left[(L + x)^2 \frac{\partial s_n}{\partial x}\right] = R \frac{\partial s_n}{\partial x} + c \frac{\partial s_{n-1}}{\partial t}, \quad n=1,2,\ldots \tag{4.1}$$

where the $s_n(x,t)$ $(n=0,1,2,\ldots)$ are assumed to satisfy the boundary conditions

$$s_n(0,t) = \sigma \qquad \text{for all } t, \tag{5}$$

$$s_n(-L(t),t) = 0 \qquad \text{for all } t. \tag{6}$$

Integrating (4) we have

$$D(L+x)^2 \frac{\partial s_o}{\partial x} - Rs_o = f_1(t) \quad . \tag{7}$$

Here, as well as in the succeeding equations, the $f_i(t)$ (i=1,2,...) will be used to denote arbitrary functions of time which will be determined by the boundary conditions (5) and (6). Multiplying both sides of (7) by the integrating factor

$$\frac{1}{D(L+x)^2}\, e^{\frac{R}{D(L+x)}}$$

and integrating, we have

$$s_o(x,t) = -\frac{f_1(t)}{R} + f_2(t)e^{\frac{-R}{D(L+x)}} \quad . \tag{8}$$

Imposing (5) and (6) on this solution we have

$$f_1(t) \equiv 0 \ , \qquad\qquad f_2(t) = \sigma\, e^{\frac{R}{DL}} \ ,$$

so that

$$s_o(x,t) = \sigma e^{\frac{R}{D}\left(\frac{1}{L} - \frac{1}{L+x}\right)} \quad . \tag{9}$$

Integrating (4.1) we have, for $s_1(x,t)$,

$$D(L+x)^2 \frac{\partial s_1}{\partial x} - Rs_1 = c \int_{-L}^{x} \frac{\partial s_o}{\partial t}\, dx + f_3(t) \quad . \tag{10}$$

The lower limit in the integral in (10) is set equal to -L merely in order to simplify the application of the boundary conditions. As with (7), we multiply both sides of (10) by the integrating factor

$$\frac{1}{D(L+x)^2}\, e^{\frac{R}{D(L+x)}}$$

and integrate, obtaining

$$s_1(x,t) = ce^{\frac{-R}{D(L+x)}} \int_{0}^{x} \frac{1}{D(L+x)^2}\, e^{\frac{R}{D(L+x)}} \int_{-L}^{x} \frac{\partial s_o}{\partial t}\, dx\, dx \ -$$

$$- \frac{f_3(t)}{R} + f_4(t)e^{\frac{-R}{D(L+x)}} \quad , \qquad (11)$$

the lower limit of the integration again being chosen in order to simplify the application of the boundary conditions. Considering the double integral in (11) we have, upon integrating by parts,

$$\int_0^x \frac{1}{D(L+x)^2} e^{D(L+x)} \int_{-L}^x \frac{\partial s_o}{\partial t} dx\, dx =$$

$$- \frac{1}{R}\left[e^{\frac{R}{D(L+x)}} \int_{-L}^x \frac{\partial s_o}{\partial t} dx\right]_0^x + \frac{1}{R}\int_0^x e^{\frac{R}{D(L+x)}} \frac{\partial s_o}{\partial t} dx. \qquad (12)$$

Substitution of (12) in (11) then gives

$$s_1(x,t) = \frac{c}{R} e^{\frac{-R}{D(L+x)}} \int_0^x e^{\frac{R}{D(L+x)}} \frac{\partial s_o}{\partial t} dx - \frac{c}{R}\int_{-L}^x \frac{\partial s_o}{\partial t} dx$$

$$+ \frac{c}{R} e^{\frac{R}{D}(\frac{1}{L} - \frac{1}{L+x})} \int_{-L}^0 \frac{\partial s_o}{\partial t} dx - \frac{f_3(t)}{R} + f_4(t)e^{\frac{-R}{D(L+x)}}. \qquad (13)$$

From (9) we have

$$\frac{\partial s_o}{\partial t} = \frac{g}{D}[(\frac{R}{L})' - \frac{R'}{L+x} + \frac{RL'}{(L+x)^2}]e^{\frac{R}{D}(\frac{1}{L} - \frac{1}{L+x})} \quad , \qquad (14)$$

in which the primes denote differentiation with respect to time. Thus the integrand $\frac{\partial s_o}{\partial t}$ in two of the integrals in (13) remains finite over the range of integration, $-L \leq x \leq 0$. Further, although the integrand in the first integral on the right hand side of (13) becomes infinite at $x = -L$,

$$\lim_{x\to -L} e^{\frac{-R}{D(L+x)}} \int_0^x e^{\frac{R}{D(L+x)}} \frac{\partial s_o}{\partial t} dx \to 0 \; .$$

Thus we may apply boundary condition (6) to (13) and obtain

$$f_3(t) \equiv 0.$$

Then applying (5) to (13) we have

$$f_4(t) = \sigma e^{\frac{R}{DL}} ,$$

so that

$$s_1(x,t) = s_o(x,t)\,[1 + \frac{c}{\sigma R}\, e^{\frac{-R}{DL}} \int_0^x e^{\frac{R}{D(L+x)}} \frac{\partial s_o}{\partial t}\, dx$$

$$+ \frac{c}{\sigma R} \int_{-L}^{0} \frac{\partial s_o}{\partial t}\, dx - \frac{c}{\sigma R}\, e^{\frac{R}{D}(\frac{1}{L+x} - \frac{1}{L})} \int_{-L}^{x} \frac{\partial s_o}{\partial t}\, dx] . \qquad (15)$$

We consider now the various terms in (15). From (14)

$$\frac{c}{\sigma R}\, e^{\frac{-R}{DL}} \int_0^x e^{\frac{R}{D(L+x)}} \frac{\partial s_o}{\partial t}\, dx = \frac{c}{DR} \int_0^x [\,(\frac{R}{L})' - \frac{R'}{L+x} + \frac{RL'}{(L+x)^2}]dx \qquad (16)$$

$$= \frac{cx}{DR}(\frac{R}{L})' + \frac{cR'}{DR} \log\left|\frac{L}{L+x}\right| + \frac{cL'}{D}(\frac{1}{L} - \frac{1}{L+x}) .$$

Also from (14) we have

$$\frac{c}{\sigma R}\, e^{\frac{R}{DL}} \int_{-L}^{0} \frac{\partial s_o}{\partial t}\, dx - \frac{c}{\sigma R}\, e^{\frac{R}{D}(\frac{1}{L+x} - \frac{1}{L})} \int_{-L}^{x} \frac{\partial s_o}{\partial t}\, dx$$

$$= \frac{c}{DR}\, e^{\frac{R}{DL}} \int_{-L}^{0} [\,(\frac{R}{L})' - \frac{R'}{L+x} + \frac{RL'}{(L+x)^2}]e^{\frac{-R}{D(L+x)}}\, dx$$

$$- \frac{c}{DR}\, e^{\frac{R}{D(L+x)}} \int_{-L}^{x} [\,(\frac{R}{L})' - \frac{R'}{L+x} + \frac{RL'}{(L+x)^2}]e^{\frac{-R}{D(L+x)}}\, dx$$

$$= \frac{c}{DR} e^{\frac{R}{DL}} \int_{-L}^{0} [(\frac{R}{L})' - \frac{R'}{L+x}] e^{\frac{-R}{D(L+x)}} dx$$

$$- \frac{c}{DR} e^{\frac{R}{D(L+x)}} \int_{-L}^{x} [(\frac{R}{L})' - \frac{R'}{L+x}] e^{\frac{-R}{D(L+x)}} dx$$

$$+ \frac{cL'}{R} e^{\frac{R}{DL}} \left[e^{\frac{-R}{D(L+x)}} \right]_{-L}^{0} - \frac{cL'}{R} e^{\frac{R}{D(L+x)}} \left[e^{\frac{-R}{D(L+x)}} \right]_{-L}^{x} .$$

Noting that the last two terms in the expression directly above cancel each other, we have

$$\frac{c}{\sigma R} \int_{-L}^{0} \frac{\partial s_c}{\partial t} dx - \frac{c}{\sigma R} e^{\frac{R}{D}(\frac{1}{L+x} - \frac{1}{L})} \int_{-L}^{x} \frac{\partial s_c}{\partial t} dx \quad \ldots$$

$$= \frac{c}{DR}(\frac{R}{L})' [e^{\frac{R}{DL}} \int_{-L}^{0} (1 - \frac{a}{L+x}) e^{\frac{-R}{D(L+x)}} dx - e^{\frac{R}{D(L+x)}} \int_{-L}^{x} (1 - \frac{a}{L+x}) e^{\frac{-R}{D(L+x)}} dx] \tag{17}$$

where

$$a = \frac{R'}{(\frac{R}{L})'} \tag{18}$$

In each of the integrals in (17) let $y = \frac{R}{D(L+x)}$ so that

$$\int_{-L}^{x} (1 - \frac{a}{L+x}) e^{\frac{-R}{D(L+x)}} dx = \frac{R}{D} \int_{\frac{R}{D(L+x)}}^{\infty} (\frac{1}{y^2} - \frac{aD}{Ry}) e^{-y} dy .$$

Integrating the first term in the integral on the right hand side of the equation directly above by parts we have

$$\int_{\frac{R}{D(L+x)}}^{\infty} \frac{e^{-y}}{y^2}\, dy = \left[-\frac{e^{-y}}{y} \right]_{\frac{R}{D(L+x)}}^{\infty} - \int_{\frac{R}{D(L+x)}}^{\infty} \frac{e^{-y}}{y}\, dy \, ,$$

so that

$$\int_{-L}^{x} \left(1 - \frac{\alpha}{L+x}\right) e^{\frac{-R}{D(L+x)}}\, dx = (L + x) e^{\frac{-R}{D(L+x)}} - \left(\alpha + \frac{R}{D}\right) \int_{\frac{R}{D(L+x)}}^{\infty} \frac{e^{-y}}{y}\, dy. \quad (19)$$

Substituting (19) in (17) we have

$$\frac{c}{\sigma R} \int_{-L}^{0} \frac{\partial s_0}{\partial t}\, dx - \frac{c}{\sigma R}\, e^{\frac{R}{D}\left(\frac{1}{L+x} - \frac{1}{L}\right)} \int_{-L}^{x} \frac{\partial s_0}{\partial t}\, dx = \frac{\alpha}{DR}\left(\frac{R}{L}\right)' \Bigg\{ - x$$

$$+ \left(\alpha + \frac{R}{D}\right)\Bigg[e^{\frac{R}{D(L+x)}} \int_{\frac{R}{D(L+x)}}^{\infty} \frac{e^{-y}}{y}\, dy - e^{\frac{R}{DL}} \int_{\frac{R}{DL}}^{\infty} \frac{e^{-y}}{y}\, dy \Bigg] \Bigg\} . \quad (20)$$

Finally, substituting (16) and (20) in (15), we have

$$s_1(x,t) = s_c(x,t) \Bigg\{ \Bigg[1 + \frac{\alpha R'}{DR} \log\left| \frac{L}{L+x} \right| + \frac{\alpha L'}{D}\left(\frac{1}{L} - \frac{1}{L+x}\right)$$

$$+ \frac{\alpha R'}{DR}\left(1 + \frac{R}{\alpha D}\right) \left[M\left(\frac{R}{D(L+x)}\right) - M\left(\frac{R}{DL}\right) \right] \Bigg] \Bigg\} , \quad (21)$$

where

$$M(x) \equiv e^{x} \int_{x}^{\infty} \frac{e^{-y}}{y}\, dy \, . \quad (22)$$

It may be noted that values of the function

$$- Ei(-x) \equiv \int_{x}^{\infty} \frac{e^{-y}}{y}\, dy$$

are tabulated (see [2].)

For the case in which the mean cross-sectional area c is a function of x the formal procedure leading to s_0 and s_1 is identical with that just given although the details of the integrations are more cumbersome. The term $D \frac{\partial}{\partial x} (x^2 \frac{\partial s}{\partial x})$ in (1) must then be modified slightly and may be obtained from Eq. (97) of reference [1].

We now consider the solution of (1) for the case in which R is constant and $q(x,t) = q(x)$ so that we may assume $s(x,t) = s(x)$. Then (1) becomes

$$Dx^2 s'' + (2Dx - R)s' = - q(x) , \tag{23}$$

in which primes denote differentiation with respect to x. The change of both independent and dependent variables given by

$$s(x) = e^{-z}\psi(z) , \qquad z = \frac{R}{2Dx} \tag{24}$$

simplifies (25) to the particularly nice form

$$\psi'' - \psi = \frac{-z^{-2}e^{z}}{D} q(\frac{R}{2Dz}) \equiv f(z) . \tag{25}$$

Using the method of variation of parameters the general solution to (25) is

$$\psi = \psi_1 \int_{\infty}^{z} \psi_2 f(z)dz + \int_{z}^{\frac{R}{2DL}} \psi_1 f(z)dz + c_1 e^{z} + c_2 e^{-z} ,$$

where c_1 and c_2 are arbitrary constants and ψ_1 and ψ_2 are two solutions to the homogeneous equation $\psi'' - \psi = 0$ such that

$$\psi_1'\psi_2 - \psi_2'\psi_1 = 1 ,$$

for which convenient choices are

$$\psi_1 = \frac{e^z}{\sqrt{2}}\ , \qquad \psi_2 = \frac{e^{-z}}{\sqrt{2}}\ ,$$

The particular choice for the limits of integration in (26) are made in order to simplify the application of the boundary conditions. Thus

$$\psi(z) = -\frac{e^z}{2D}\int_{\infty}^{z} z^{-2}q\left(\frac{R}{2Dz}\right)dz - \frac{e^{-z}}{2D}\int_{z}^{\frac{R}{2DL}} z^{-2}e^{2z}q\left(\frac{R}{2Dz}\right)dz + c_1e^z + c_2e^{-z}. \tag{26.1}$$

Substituting (24) in (26.1) we now have

$$s(x) = \frac{1}{R}\int_{0}^{x} q(x)dx + \frac{1}{R}\, e^{\frac{-R}{Dx}}\int_{x}^{L} e^{\frac{R}{Dx}}q(x)dx + c_1 + c_2e^{\frac{-R}{Dx}} \tag{27}$$

We consider two special cases, namely

(A) $q(x) = Q\delta(x - x_o)$ in which Q is a constant, $0 < x_o < L$. This choice for q(x) should apply to the introduction of some solute at one point ($x = x_o$) in the estuary. Q is then the rate of discharge of this solute into the estuary (of dimensions MT^{-1}).

(B) $q(x) = q$ in which q is constant over the range $0 < x < L$. This choice for q(x) corresponds to introduction of solute uniformly over the entire effective length of the estuary and might apply in the investigation of ground seepage. The total rate of discharge of the solute into the estuary over the range $0 < x < L$ is then qL.

For q(x) as given in (A), (27) becomes

$$s(x) = \frac{Q}{R}\int_{0}^{x} \delta(x - x_o)dx + \frac{Q}{R}\, e^{\frac{-R}{Dx}}\int_{x}^{L} e^{\frac{R}{Dx}}\,\delta(x - x_o)dx + c_1 + c_2e^{\frac{-R}{Dx}}.$$

Thus if we impose the boundary conditions $s(0) = s(L) = 0$ ($s(0)$ being set equal to zero since there is no mechanism to carry the solute upstream of the point $x = 0$, at which point the motion in the estuary due to the tides is zero; and $s(L)$ being set equal to zero because of the essentially infinite reservoir provided by the ocean at the mouth of the estuary) then

$$c_1 = 0 , \qquad c_2 = - \frac{Q}{R} e^{\frac{R}{DL}} ,$$

so that

$$s(x) = \frac{Q}{R} \int_0^x \delta(x - x_o)dx + \frac{Q}{R} e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}} \delta(x - x_o)dx - \frac{Q}{R} e^{\frac{R}{D}(\frac{1}{L} - \frac{1}{x})} \tag{28}$$

which may be written in the form

$$\begin{aligned} s(x) &= \frac{Q}{R}[e^{\frac{R}{D}(\frac{1}{x_o} - \frac{1}{x})} - e^{\frac{R}{D}(\frac{1}{L} - \frac{1}{x})}] , & 0 \leqq x < x_o \\ s(x) &= \frac{Q}{R}[1 - e^{\frac{R}{D}(\frac{1}{L} - \frac{1}{x})}] & x_o < x \leqq L \end{aligned} \tag{29}$$

For $q(x)$ as given in (B), (27) becomes

$$s(x) = \frac{qx}{R} + \frac{q}{R} e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}} dx + c_1 + c_2 e^{\frac{-R}{Dx}} .$$

Again, imposing the boundary conditions $s(0) = s(L) = 0$ we have

$$c_1 = 0 \qquad \text{since} \qquad \lim_{x \to 0} e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}} dx = 0$$

and

$$c_2 = - \frac{qL}{R} e^{\frac{R}{DL}}$$

so that

$$s(x) = \frac{qx}{R} - \frac{qL}{R}\, e^{\frac{R}{D}\left(\frac{1}{L} - \frac{1}{x}\right)} + \frac{q}{R}\, e^{\frac{-R}{Dx}} \int_x^L e^{\frac{R}{Dx}}\, dx \,. \tag{30}$$

If we let $y = \frac{R}{Dx}$ in the integral in (30) then we have

$$\begin{aligned} \int_x^L e^{\frac{R}{Dx}}\, dx &= \frac{R}{D} \int_{\frac{R}{DL}}^{\frac{R}{Dx}} \frac{e^y}{y^2}\, dy \\ &= Le^{\frac{R}{DL}} - xe^{\frac{R}{Dx}} + \frac{R}{D} \int_{\frac{R}{DL}}^{\frac{R}{Dx}} \frac{e^y}{y}\, dy \end{aligned} \tag{31}$$

after integrating by parts. Substituting (31) in (30) we have

$$s(x) = \frac{q}{D}\, e^{\frac{-R}{Dx}} \int_{\frac{R}{DL}}^{\frac{R}{Dx}} \frac{e^y}{y}\, dy \,. \tag{32}$$

It may be interesting to compare the solute distributions that result for $q(x)$ as given in (A) and (B) respectively, assuming that the rates of discharge of solute throughout the entire effective length of the estuary are equal for the two cases, i.e., $qL = Q$. In Fig. 1 Rs/Q as a function of x/L is plotted for $R/DL = 0.8$, the value given for the Raritan River in a paper by Arons and Stommel [3]. The constant which they denote by F and call the flushing number is equal to R/DL in the notation used here. Values of the integral appearing in (32) are to be found in reference [2].

Figure 1 presents two curves for the case in which $q(x) = Q\delta(x - x_o)$; in one $x_o/L = 0.4$ and in the other $x_o/L = 0.8$. It may be noted in particular that the solute density is appreciably greater over most of the estuary for the case in which $x_o/L = 0.4$. It is to be hoped that this sensitivity of the solute density upstream of the external source to the position of the source may be used to check experimentally the validity of the solute diffusion equation (1) which we have been using.

From (29) we may note that the maximum value of $s(x)$ occurs at $x = x_o$ in the case in which $q(x) = Q\delta(x - x_o)$. In Fig. 2, $Rs(x_o)/Q$ as a function of x_o/L is plotted. A possible application of the data given in Fig. 2 is the determination of the maximum distance from the mouth of an estuary at which a known source of pollution may be allowed to discharge into an estuary if specified limits are placed on the permissible density of pollution in the estuary.

References

[1] A theory of tidal mixing in an essentially vertically homogeneous estuary, by L. C. Maximon and G. W. Morgan, Technical Report No. 2, Nonr-56202 (NR-083-067). Note, specifically, Eq. (119).

[2] Tables of Sine, Cosine and Exponential Integrals, vols I and II. Federal Works Agency Work Projects Administration (1940).

[3] A mixing-length theory of tidal flushing, by A. B. Arons and H. Stommel, Trans. Amer. Geophys. Union, 32, no. 3 (1951).

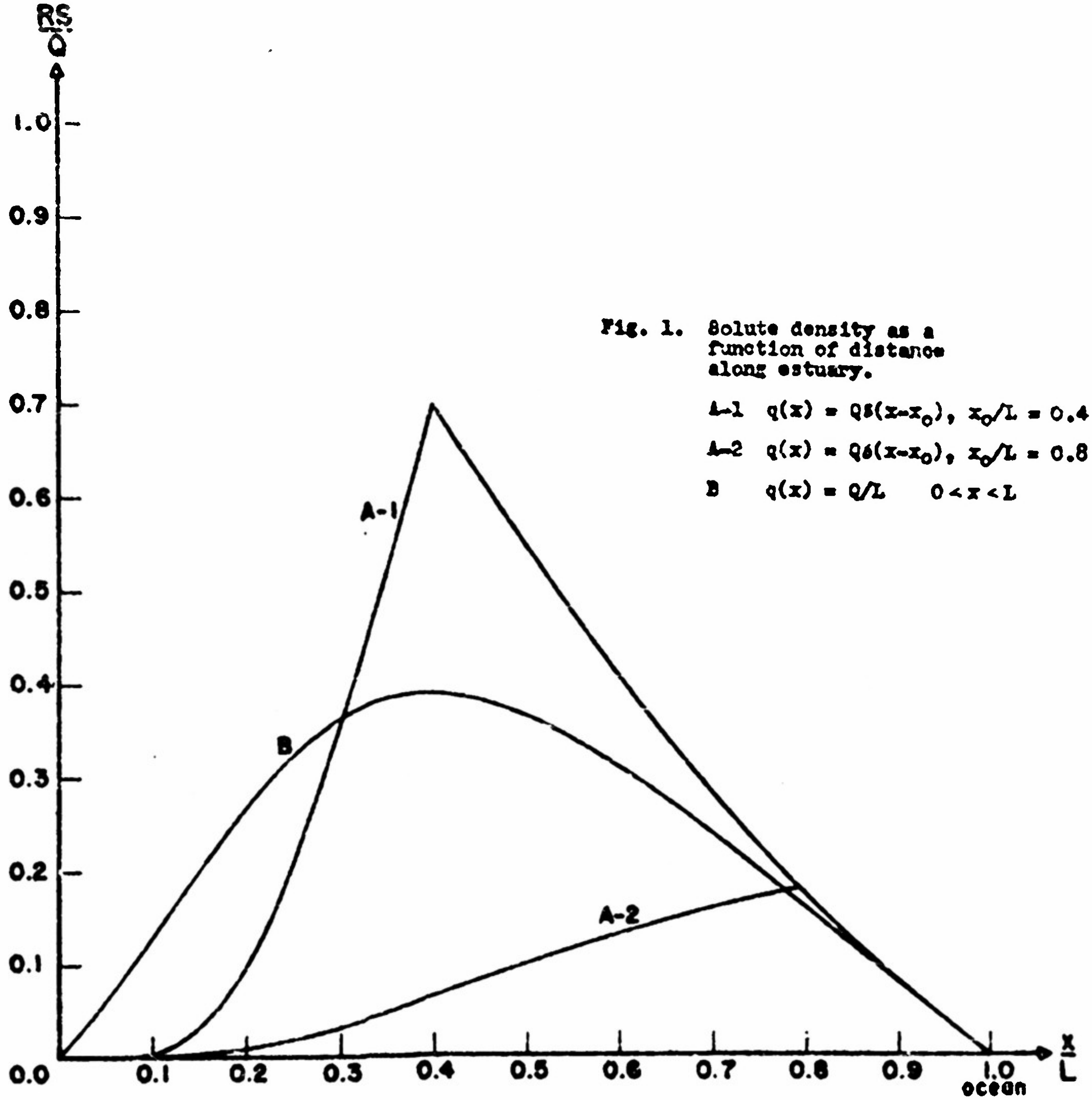

Fig. 1. Solute density as a function of distance along estuary.

A-1 $q(x) = Q\delta(x-x_0)$, $x_0/L = 0.4$

A-2 $q(x) = Q\delta(x-x_0)$, $x_0/L = 0.8$

B $q(x) = Q/L \quad 0 < x < L$

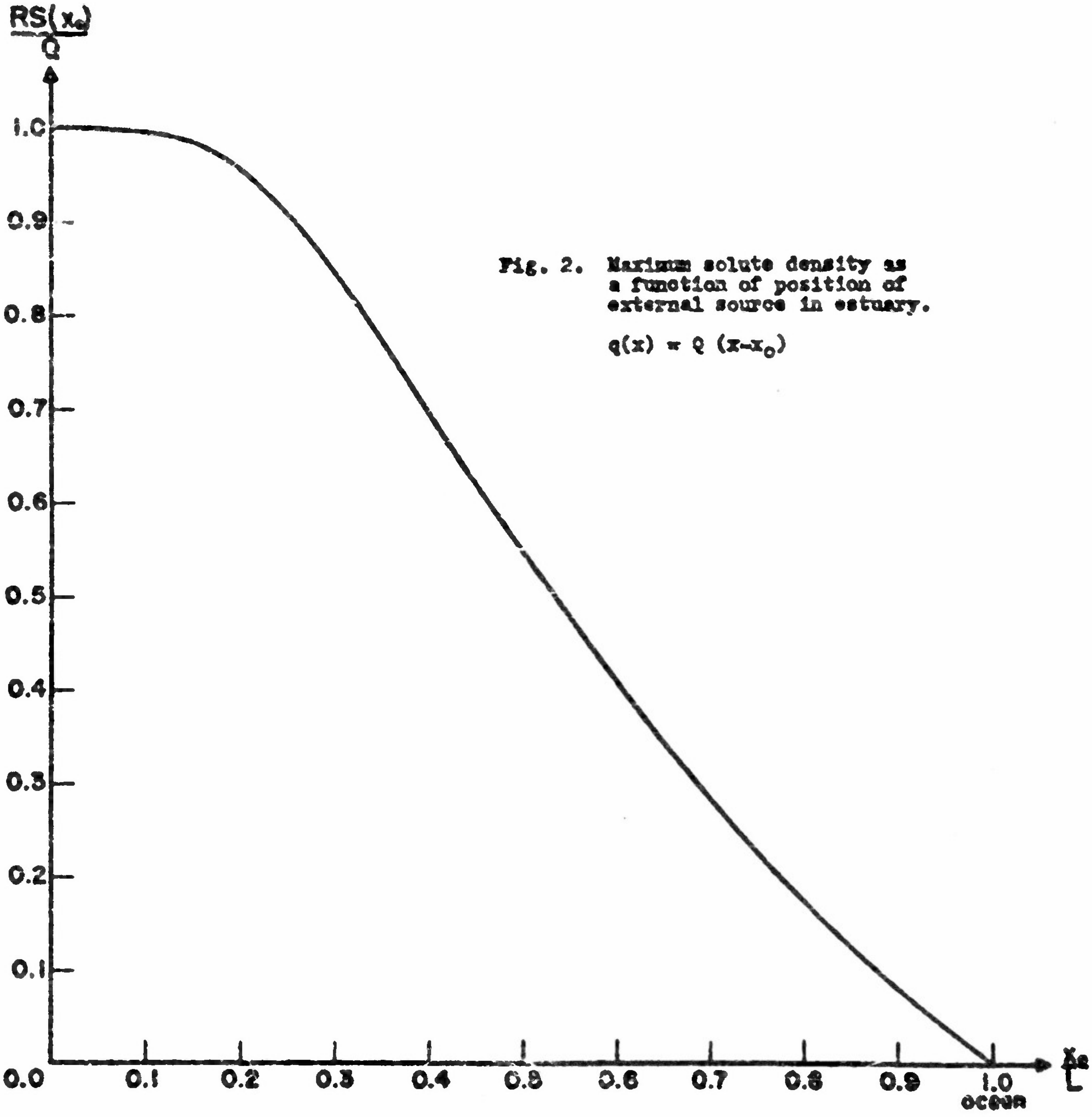

Fig. 2. Maximum solute density as a function of position of external source in estuary.

$q(x) = Q\ (x-x_0)$